NUMERICAL COMPUTATIONAL TECHNIQUES

UNIT-1

SOLUTIONS OF EQUATIONS AND EIGEN VALUE PROBLEMS

Solution of Algebraic and Transcendental equations:

The equation of the form $f(x)=0$ are called algebraic equations if $f(x)$ is purely a polynomial in x.

For example $x^4 - x = 10$ and $2x^3 - 3x^2 - x + 5 = 0$ are algebraic equations.

If $f(x)$ also contains trignometric, logarithmic, exponential function etc.., then the equation $f(x)=0$ is known as transcendental equation.

For example: $x \log x - 1.2 = 0$, $xe^x - \cos x = 0$.

Location of Roots:

If $f(x)$ is a continuous function in the interval (a,b) and if $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x)=0$ has atleast one real root lying in the interval (a,b).

The following iterative method is used to solve the equation $f(x)=0$:

Newton-Raphson method (or) Newton's method.

This method starts with an initial approximation to the root of an equation, a better and closer approximation to the root can be found by using an iterative process.

Newton-Raphson iterative formula is

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2\ldots}$$

* Order of convergence of N.R. method is 2.
* Convergence condition for N.R. method is

$$|f(x)\, f''(x)| < |f'(x)|^2.$$

PROBLEMS :

1) Show that the N.R. formula to find $\sqrt{a}$ can be expressed in the form $x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right]$, $n = 0, 1, 2 \ldots$

Solution :-

If $x = \sqrt{a}$, then $x^2 - a = 0$.

Let $f(x) = x^2 - a$

$f'(x) = 2x$.

By N.R. formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{x_n^2 - a}{2x_n}\right]$$

$$= \frac{2x_n^2 - x_n^2 + a}{2x_n}$$

$$= \frac{x_n^2 + a}{2x_n}$$

$$= \frac{1}{2}\left[\frac{x_n^{\cancel{2}}}{\cancel{x_n}} + \frac{a}{x_n}\right]$$

$$\boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right], \quad n = 0, 1, 2 \ldots}$$

② Using Newton's iterative method find the root between 0 and 1 of $x^3 = 6x - 4$ correct to two decimal places.

Solution :- Let $f(x) = x^3 - 6x + 4$; $f'(x) = 3x^2 - 6$

$f(0) = 4 = +ve$

$f(1) = 1 - 6 + 4 = 5 - 6 = -1 = -ve$

$\therefore$ a root lies between 0 and 1.

$|f(1)| < |f(0)|$

$\therefore$ This root is nearer to 1.

By Newton's iterative formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1 \ldots$$

Take $x_0 = 1$.

First approximation

Put $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)} = 1 - \left[\frac{(-1)}{3(1)^2 - 6}\right]$$

$$= 1 - \left(\frac{-1}{-3}\right) = 1 - \frac{1}{3}$$

$$= 0.666$$

$= 0.67$ (correct to two decimal places)

Second approximation:

Put $n = 1$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.67 - \frac{f(0.67)}{f'(0.67)}$$

$$= 0.67 - \left[\frac{(0.67)^3 - 6(0.67) + 4}{3(0.67)^2 - 6}\right]$$

$$= 0.67 - \left[\frac{0.28}{-4.65}\right]$$

$$= 0.67 + \frac{0.28}{4.65} = 0.73$$

Third approximation:

Put $n = 2$.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73 - \frac{f(0.73)}{f'(0.73)}$$

$$= 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3(0.73)^2 - 6}\right]$$

$$= 0.73 - \left[\frac{0.009}{-4.4013}\right]$$

$$= 0.73 + \frac{0.009}{4.4013}$$

$$= 0.7320$$

$= 0.73$ (correct to two decimal places)

Here $x_2 = x_3 = 0.73$

$\therefore$ The root is 0.73 correct to two decimal places.

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3. Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to four decimal places.

Solution: Let $f(x) = 3x - \cos x - 1$

$$f'(x) = 3 + \sin x$$

$f(0) = 0-1-1 = -2 = -ve$

$f(1) = 3-\cos 1 - 1 = 2-\cos 1 = 1.4597 = +ve$

$\therefore$ a root lies between 0 and 1.

$|f(1)| < |f(0)|$

Hence the root is nearer to 1.

Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2\ldots$$

Let $x_0 = 0.6$

Put $n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.6 - \frac{f(0.6)}{f'(0.6)}$$

$$= 0.6 - \left[\frac{3(0.6) - \cos(0.6) - 1}{3+\sin(0.6)}\right]$$

$$= 0.6 - (-0.007101)$$

$$= 0.607108 = 0.6071 \quad \text{(4 decimal places)}$$

Put $n=1$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.607108 - \left[\frac{3(0.607108) - \cos(0.607108) - 1}{3+\sin(0.607108)}\right]$$

$$= 0.607108 - 0.000006$$

$$= 0.607102 = 0.6071 \quad \text{(4 decimal places)}$$

~~Root ans~~

Here $x_1 = x_2 = 0.6071$.

$\therefore$ The required root is 0.6071.

④ ⑤

Find a root of $x\log_{10}x - 1.2 = 0$ by N.R. method correct to three decimal places.

Solution :-

Let $f(x) = x\log_{10}x - 1.2$

$$f'(x) = \log_{10}x + x\left(\frac{1}{x}\right)\log_{10}e$$

$$= \log_{10}x + \log_{10}e.$$

$$\therefore f'(x) = \log_{10}x + 0.4343$$

$f(0) = 0 - 1.2 = -1.2 = -ve$

$f(1) = \log_{10}1 - 1.2 = -1.2 = -ve$

$f(2) = 2\log_{10}2 - 1.2 = -0.598 = -ve$

$f(3) = 3\log_{10}3 - 1.2 = 0.231 = +ve$

$|f(3)| < |f(2)|$

$\therefore$ a root lies between 2 and 3 and also it is nearer to 3.

Let $x_0 = 2.7$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

Put $n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.7 - \frac{f(2.7)}{f'(2.7)}$$

$$= 2.7 - \left[\frac{(2.7)\log_{10}2.7 - 1.2}{\log_{10}e + \log_{10}2.7}\right]$$

$$= 2.7 - \left[\frac{-0.035}{0.867}\right]$$

$$= 2.7 + \frac{0.035}{0.867} = 2.740$$

Put $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.740 - \frac{f(2.740)}{f'(2.740)}$$

$$= 2.74 - \left[\frac{(2.740)\log_{10} 2.740 - 1.2}{\log_{10} e + \log_{10} 2.740}\right]$$

$$= 2.74 - \left[\frac{-0.0006}{0.872}\right]$$

$$= 2.74 + \frac{0.0006}{0.872} = 2.741.$$

Put $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.741 - \frac{f(2.741)}{f'(2.741)}$$

$$= 2.741 - \left[\frac{(2.741)\log_{10} 2.741 - 1.2}{\log_{10} e + \log_{10} 2.741}\right]$$

$$= 2.741 - \frac{0.003}{0.872}$$

$$= 2.741$$

Here $x_2 = x_3 = 2.741$

$\therefore$ The required root is 2.741.

(5) Find the iterative formula for finding the value of $\frac{1}{N}$ where N is a real number, using Newton's method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Solution :-

Let $x = \frac{1}{N}$

i.e., $N = \frac{1}{x}$

Let $f(x) = \frac{1}{x} - N$; $f'(x) = -\frac{1}{x^2}$

We know that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

$$= x_n - \left[\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right]$$

$$= x_n + x_n^2\left[\frac{1}{x_n} - N\right]$$

$$= x_n + \frac{x_n^{\cancel{2}}}{\cancel{x_n}} - Nx_n^2$$

$$= x_n + x_n - Nx_n^2$$

$$= 2x_n - Nx_n^2 = x_n[2 - Nx_n]$$ is the iterative formula.

To find $\frac{1}{26}$, take $N = 26$.

Let $x_0 = 0.04$ $\quad \left[\because \frac{1}{25} = 0.04\right]$

$$x_{n+1} = x_n[2 - Nx_n]$$

Put $n = 0$;

$$x_1 = x_0[2 - Nx_0] = 0.04[2 - 26(0.04)]$$
$$= 0.0384$$

Put $n = 1$;

$$x_2 = x_1[2 - 26x_1] = 0.0384[2 - 26(0.0384)]$$
$$= 0.0385$$

Put $n = 2$;

$$x_3 = x_2[2 - 26x_2]$$
$$= 0.0385[2 - 26(0.0385)] = 0.0385$$

Here $x_2 = x_3 = 0.0385$

Hence the value of $\frac{1}{26} = 0.0385$ //.

Homework problems :

1) Find the positive root of $x^4 - x - 10 = 0$ by Newton's method, correct to four decimal places.

Ans: 1.8556

2) Find by Newton's method, the real positive root of $x = \cos x$, correct to three decimal places.

Ans: 0.739

(3). Obtain Newton's iterative formula for finding $\sqrt{N}$ where N is a positive real number. Hence evaluate $\sqrt{142}$.

Ans: 11.9164.